

Multiplicative Lévy processes: Itô versus Stratonovich interpretation

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Langevin equation with a multiplicative stochastic force is considered. That force is uncorrelated, it has the Lévy distribution and the power-law intensity. The Fokker-Planck equations, which correspond both to the Itô and Stratonovich interpretation, are presented. They are solved for the case without drift and for the harmonic oscillator potential. The variance is evaluated; it is always infinite for the Itô case whereas for the Stratonovich one it can be finite and rise with time slower than linearly, which indicates subdiffusion. Analytical results are compared with numerical simulations.

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I. INTRODUCTION

The Langevin formalism was introduced to describe motion of a particle which was subjected both to a Newtonian deterministic force and to the irregular influence of a bath of small molecules (the Brownian motion). That random force is uncorrelated and usually it is taken in the Gaussian form. Importance and wide applicability of the Gaussian distribution in the statistical physics follows from its stability: it constitutes an attractor in the functional space. According to the central limit theorem, a superposition of distributions with a finite variance leads to the Gaussian and its second moment is proportional to the time. However, many phenomena cannot be described in this way. Variance may depend on time stronger or weaker than linearly; as a consequence, the diffusion is anomalous [1]. Moreover, one can frequently encounter in nature systems far from thermal equilibrium, for which moments, in particular the variance, are divergent [2–6]. From that point of view, the Lévy process, which is stable but may be characterized by divergent moments, is an important generalization of the Gaussian process.

Divergent moments of the stochastic driving force can be attributed to nonhomogeneous structure of the environment, e.g., fractal or multifractal, which produces long-range correlations. Lévy statistics can emerge from the temporal nature of the underlying process due to a subordination to the ordinary Brownian motion, which is highly inhomogeneous [7]. The medium nonhomogeneity can enter the Langevin equation via the deterministic force. It is even possible to model random effects in this way (the quenched disorder [1,8]). However, there are also processes for which the fluctuations of the stochastic force directly depend on the state of the system. The autocatalytic chemical reaction, in which the production of a molecule is enhanced by the presence of the molecules of the same type, can serve as an example [9]. As a result, the fluctuating term in the Langevin equation is multiplied by some function of the macroscopic variables (the multiplicative noise). Some physical problems require taking into account both additive and multiplicative noise [10,11].

In this paper, we consider the Langevin equation with the multiplicative noise,

$$\dot{x} = F(x) + G(x)\eta(t), \quad (1)$$

in which $F(x)$ is the deterministic force and the uncorrelated stochastic force $\eta(t)$ possesses the Lévy distribution. The

case $G(x)=\text{const}$ has been extensively studied. It has been demonstrated [12] that both the force-free system and that driven by the linear force are described by the Lévy distribution and then the variance is divergent. The strongly nonlinear force, however, is able to confine Lévy flights and make the variance finite [13,14]. Also problems with more complicated forms of $F(x)$ were considered in context of Eq. (1) with the Lévy noise. They involve the barrier penetration [15] and escape from the potential well [16], as well as the transport in a Lévy ratchet [17]. The general case, with multiplicative noise, can be regarded as a result of the adiabatic elimination of fast variables for nonlinear processes with additive fluctuations. Equation (1) can also be directly applied to model specific phenomena with fluctuations which are characterized by long tails in the distribution and a variable, in particular power-law, intensity. It is the case, for example, in the field of finance, where Eq. (1) could be a natural generalization of the Black-Scholes equation [18,19].

In the general case of variable $G(x)$ we encounter the well-known problem of interpretation of the noise term in Eq. (1) and then of the Stieltjes integral $\int_0^t G[x(\tau)]d\eta(\tau)$ which, in this form, is meaningless [20,21]. We must decide whether $G[x(\tau)]$ is calculated before the noise acts, or after that. The former case corresponds to the Itô interpretation [22]

$$\int_0^t G[x(\tau)]d\eta(\tau) = \sum_{i=1}^n G[x(t_{i-1})][\eta(t_i) - \eta(t_{i-1})], \quad (2)$$

where the interval $(0, t)$ has been divided in n subintervals ($n \rightarrow \infty$). The above integral does not obey standard rules of the calculus, in particular the chain rule. Those rules are satisfied by the stochastic calculus introduced by Stratonovich. The stochastic integral [23] in this interpretation includes the process value both at the beginning and at the end of each subinterval,

$$\int_0^t G[x(\tau)]d\eta(\tau) = \sum_{i=1}^n G\left[\frac{x(t_{i-1}) + x(t_i)}{2}\right][\eta(t_i) - \eta(t_{i-1})]. \quad (3)$$

From the mathematical and technical point of view, the Itô interpretation is easier to apply e.g., in the perturbation theory [20]. It is a suitable choice if the noise consists of

clearly separated pulses, e.g., for a continuous description of integer processes. If, on the other hand, a system has some finite correlations and the white noise is only an approximation, the Stratonovich interpretation is more appropriate. It is the case if the noise has an external source, i.e., the noise source is not influenced by the system itself and it is possible, in principle, to turn off the noise [24]. Possibility of applying standard rules of the calculus is an important technical advantage of the Stratonovich interpretation. It makes possible to solve the Langevin equation exactly for some nonlinear models with the multiplicative Gaussian white noise [25]. Physical predictions which follow from the Langevin equation with the noise in both interpretations can be qualitatively different. It is the case, for example, for the Ginzburg-Landau model with external multiplicative fluctuations, which describes noise-induced phase transitions caused by short-term instabilities of the disordered phase. The system exhibits that transition if one interprets the noise in the Stratonovich sense, but not if one uses the Itô interpretation [26]. We demonstrate in this paper that predictions of both formalisms are different also for the Lévy diffusion process.

The aim of this paper is to study the Langevin equation (1) with the multiplicative noise, which is given by the Lévy distribution and the algebraic $G(x)$. The Fokker-Planck equations (FPE) for both interpretations are presented in Sec. II. In Sec. III, the equation without external potential is solved and the case of the harmonic oscillator potential is discussed in Sec. IV. Results are compared with numerical simulations in Sec. V.

II. FOKKER-PLANCK EQUATIONS

In this section, we derive equations for the probability density distribution, which correspond to Eq. (1) in both interpretations. The noise $\eta(t)$ is the symmetric Lévy stable distribution, defined by the following expression:

$$p_\eta(x) = \sqrt{2/\pi} \int_0^\infty \exp(-K^\mu k^\mu) \cos(kx) dk \quad (4)$$

in terms of the order parameter $\mu(0 < \mu \leq 2)$ and the generalized diffusion constant $K > 0$. If $\mu < 2$, p_η has long tails: $p_\eta \sim |x|^{-\mu-1}$ for $|x| \rightarrow \infty$. The general Lévy processes can be defined by the Lévy-Khintchine formula which expresses the characteristic function in terms of the Lévy measure $\nu(x)$ [27]. In the symmetric and non-Gaussian case it reads

$$\ln \tilde{p}_t(k) = -t \left[\int_{|x| \geq 1} (1 - e^{ikx}) \nu(x) dx + \int_{|x| < 1} (1 - e^{ikx} + ikx) \nu(x) dx \right] \quad (5)$$

and $\nu(x) = |x|^{-\mu-1}$ corresponds to the stable process.

In the Itô interpretation, the probability density distribution is given by the fractional Fokker-Planck equation with variable diffusion coefficient [15,28]

$$\frac{\partial}{\partial t} p_I(x,t) = - \frac{\partial}{\partial x} F(x) p_I(x,t) + K^\mu \frac{\partial^\mu}{\partial |x|^\mu} [|G(x)|^\mu p_I(x,t)], \quad (6)$$

where the Riesz-Weyl fractional operator [29] is defined in terms of the Fourier transform: $\frac{\partial^\mu}{\partial |x|^\mu} = \mathcal{F}^{-1}(-|k|^\mu)$. Equations of the form (6) can describe also jumping processes. For example, the fractional equation with the variable diffusion coefficient follows from the master equation which models the thermal activation of particles within the folded, heterogeneous polymers [30]; variability of the diffusion coefficient results there from the intrinsic potential of the monomer. Moreover, Markovian versions of the continuous time random walk (CTRW) produce equations of the form (6) in the diffusion limit. In particular, the coupled CTRW model with a variable jumping rate, which describes Lévy flights in nonhomogeneous media, involves the fractional FPE [31] in the form (6). The diffusion term in Eq. (6) is, in this case, the jumping rate. The drift term may also appear if we allow for a nonvanishing mean of the Lévy distribution. The master equation describes also systems which are characterized by the internal noise. Those fluctuations emerge in systems of discrete particles and they are an inherent part of the very mechanism by which the state of the system evolves [24]. A precise form of the deterministic equation does not exist since it is impossible for systems with that noise to eliminate the fluctuations. Consequently, the master equation describes the evolution of the entire system as a stochastic process.

The Stratonovich interpretation of the stochastic integral, Eq. (3), means that rules of the calculus—the chain rule and the ordinary variable transformation formula—can be applied. The stochastic variable can be determined by a stochastic equation with the additive noise which results from that with the multiplicative noise, obtained by a variable transformation. The above property of the stochastic integral can be proved on the assumption that the noise has a finite variance [20,21]. If, in addition, trajectories are continuous (Lindeberg's condition), rules of the ordinary calculus apply to the Fisk-Stratonovich integral and one obtains a relation between integrals defined by Eqs. (2) and (3): they differ only by a simple additive term [27]. As regards the Lévy stable processes, they can be approximated by processes with the finite variance by introducing a cutoff in the Lévy measure $\nu(x)$ in Eq. (5) (truncated Lévy flights). Such an approximation is very accurate, also for a large value of the argument [32]. In Sec. V, we will present numerical examples which confirm applicability of variable transformation rules of the ordinary calculus for the Lévy stable processes.

Knowing the variable transformation rules, it is possible to transform Eq. (1) to a new Langevin equation, which contains the additive noise, instead of the multiplicative one [33]. For that purpose we introduce a new variable y and reduce Eq. (1) to the form

$$\dot{y} = \hat{F}(y) + \eta(t), \quad (7)$$

where the transformation reads

$$y(x) = \int_0^x \frac{dx'}{KG(x')}, \quad \hat{F}(y) = F[x(y)] \frac{dy}{dx}. \quad (8)$$

The corresponding FPE is of the form

$$\frac{\partial}{\partial t} p_S(y, t) = - \frac{\partial}{\partial y} \hat{F}(y) p_S(y, t) + \frac{\partial^\mu}{\partial |y|^\mu} p_S(y, t). \quad (9)$$

After solving the above equation, the solution of the original Eq. (1) follows from the probability conservation rule,

$$p_S(x, t) = p_S(y(x), t) \frac{dy}{dx}. \quad (10)$$

For the Gaussian case, $\mu=2$, Eq. (9) can be easily expressed in terms of the original variable x and a direct relation between $p_S(x, t)$ and $p_I(x, t)$ can be established. The difference between Fokker-Planck equations for both interpretations resolves itself to the additional drift, $K^2 G(x) G'(x)$, called “spurious” or “noise-induced” drift.

In the following, we solve Eq. (1) for two cases: without external potential and with the linear drift, on the assumption of both interpretations of the stochastic equation. We assume the diffusion coefficient in the algebraic form

$$G(x) = |x|^{-\theta \mu}. \quad (11)$$

Results can be generalized to other forms of $G(x)$ by applying the method from Ref. [34].

III. FORCE-FREE CASE

We begin with the case of the Itô interpretation. Equation (6) with $F(x)=0$ becomes the fractional diffusion equation with the variable diffusion coefficient,

$$\frac{\partial p_I(x, t)}{\partial t} = K^\mu \frac{\partial^\mu [|x|^{-\theta} p_I(x, t)]}{\partial |x|^\mu}. \quad (12)$$

The above equation results not only from the Langevin equation with the multiplicative noise. It constitutes the small wave number limit (the diffusion or fluid limit) of the master equation for a jumping process in the framework of the coupled CTRW [31]. That process is defined in terms of two probability distributions: of the jumping size, in the Lévy form, and the Poissonian, position-dependent waiting time distribution. The diffusion coefficient $|G(x)|^\mu$ in Eq. (6) is then the jumping rate and the parameter θ governs the transport speed. In particular, for $\mu=2$, when the variance exists, Eq. (12) describes the anomalous diffusion process: subdiffusion for $\theta > 0$, enhanced diffusion for $\theta < 0$, and the normal diffusion for $\theta=0$. The same classification holds also for $\mu < 2$ when we introduce fractional moments [31].

Equation (12) can be solved in the diffusion limit by applying the Fox functions formalism [35–37]. Details of the derivation are presented in Refs. [31, 38]. The Fox functions are well suited for problems which involve Lévy processes since any Lévy distribution, both symmetric and asymmetric, can be expressed as the function $H_{2,2}^{1,1}(x)$ [39]. Moreover, due to the multiplication rule, the term $|x|^{-\theta} p_I(x, t)$ in Eq. (12) can be easily evaluated and it produces the Fox function of the

same order, only the coefficients are shifted. Those properties of Eq. (12) suggest the method of solution: we assume the solution in the scaling form which involve the Fox function,

$$p_I(x, t) = Na(t) H_{2,2}^{1,1} \left[a(t) |x| \left| \begin{matrix} (a_1, A_1), (a_2, A_2) \\ (b_1, B_1), (b_2, B_2) \end{matrix} \right. \right], \quad (13)$$

where N is the normalization constant, and try to adjust the coefficients, as well as to derive the function $a(t)$. To implement the approximation of small k , we pass to the Fourier space, in which Eq. (12) reads

$$\frac{\partial}{\partial t} \tilde{p}_I(k, t) = -K^\mu |k|^\mu \mathcal{F}_c[|x|^{-\theta} p_I(x, t)]. \quad (14)$$

According to the well-known formula, the Fourier transform from the Fox function is also the Fox function but of higher order: $\mathcal{F}_c[H_{2,2}^{1,1}(x)] = H_{3,4}^{2,1}(k)$. Then we expand the Fox functions, which correspond to $\tilde{p}_I(k, t)$ and $\mathcal{F}_c[|x|^{-\theta} p_I(x, t)]$, in the fractional powers of k ; terms of the order $|k|^{2\mu+\theta}$ and higher are neglected. Both sides of Eq. (14) can be adjusted only if all terms except k^0 and $|k|^\mu$ vanish. We can eliminate adverse terms by a proper choice of the Fox function coefficients. Inserting the coefficient values, determined in this way, to Eq. (12), yields the solution in the form

$$p_I(x, t) = Na(t) H_{2,2}^{1,1} \left[a(t) |x| \left| \begin{matrix} \left(1 - \frac{1-\theta}{\mu+\theta}, \frac{1}{\mu+\theta} \right), (a_2, A_2) \\ (b_1, B_1), \left(1 - \frac{1-\theta}{2+\theta}, \frac{1}{2+\theta} \right) \end{matrix} \right. \right]; \quad (15)$$

the initial condition is $p_I(x, 0) = \delta(x)$. Moreover we assume $\mu + \theta > 0$. The coefficients (a_2, A_2) and (b_1, B_1) are essentially arbitrary and one needs additional requirements to settle them. Expansion of the Fox function in Eq. (15) yields $p_I(x, t) \sim |x|^{b_1/B_1} (x \rightarrow 0)$. It explains why the coefficients (b_1, B_1) have not been determined: since small $|k|$ corresponds to large $|x|$, the diffusion approximation does not cover the region of small $|x|$. In the following, we assume $b_1 = \theta$ and $B_1 = 1$, which choice is exact for CTRW [38]. The coefficients (a_2, A_2) , in turn, correspond to the shape of the distribution in the limit $\mu \rightarrow 2$ [38]. From Eq. (14) follows a simple differential equation for $a(t)$ which yields $a(t) \sim t^{-1/(\mu+\theta)}$. The asymptotic expansion of Eq. (15) yields $p_I(x, t) \sim t^{\mu/(\mu+\theta)} |x|^{-1-\mu}$. Therefore, we obtain the Lévy process which has the same distribution as the driving process, Eq. (4). The variance and all higher moments are divergent.

For the Stratonovich interpretation, we introduce the new variable y , according to Eq. (8),

$$y(x) = \frac{\mu}{K(\mu+\theta)} |x|^{(\mu+\theta)/\mu} \operatorname{sgn} x, \quad (16)$$

which ends in Langevin equation with the additive noise. Then Eq. (9) reads

$$\frac{\partial}{\partial t} p_S(y, t) = \frac{\partial^\mu}{\partial |y|^\mu} p_S(y, t). \quad (17)$$

It can be solved exactly and the solution expressed in the form [40,41]

$$p_S(y, t) = \frac{1}{\mu |y|} H_{2,2}^{1,1} \left[\frac{|y|}{t^{1/\mu}} \left| \begin{array}{l} (1, 1/\mu), (1, 1/2) \\ (1, 1), (1, 1/2) \end{array} \right. \right], \quad (18)$$

which corresponds to the symmetric Lévy process $y(t)$. The inverse transformation to the original variable yields

$$p_S(x, t) = \frac{\mu + \theta}{\mu^2 |x|} H_{2,2}^{1,1} \left[\frac{|x|^{1+\theta/\mu}}{(1 + \theta/\mu)(K^\mu t)^{1/\mu}} \left| \begin{array}{l} (1, 1/\mu), (1, 1/2) \\ (1, 1), (1, 1/2) \end{array} \right. \right]. \quad (19)$$

The asymptotic expansion of the Fox function in Eq. (19) is given by the expression $p_S(x, t) \sim t^{\mu/(\mu+\theta)} |x|^{-1-\mu-\theta} (|x| \rightarrow \infty)$. It differs considerably from the Itô result: the shape of the tail depends not only on the order parameter of the driving process, μ , but also the θ dependence emerges. As a result, the variance may not be divergent. We can express the variance by Mellin transform of the Fox function, $\chi(s)$, and evaluate it by simple algebra,

$$\begin{aligned} \langle x^2 \rangle &= 2 \int_0^\infty x^2 p(x, t) dx = 2 \left[K \left(\frac{\theta}{\mu} + 1 \right) \right]^{2\mu/(\mu+\theta)} \\ &\times \int_0^\infty y^{2\mu/(\mu+\theta)} p(y, t) dy = \frac{2}{\mu} \\ &\times \left[K \left(\frac{\theta}{\mu} + 1 \right) \right]^{2\mu/(\mu+\theta)} t^{2/(\mu+\theta)} \chi \left(-\frac{2\mu}{\mu+\theta} \right) \\ &= -\frac{2}{\pi\mu} \left[K \left(\frac{\theta}{\mu} + 1 \right) \right]^{2\mu/(\mu+\theta)} \Gamma \left(-\frac{2}{\mu+\theta} \right) \\ &\times \Gamma \left(1 + \frac{2\mu}{\mu+\theta} \right) \sin \left(\frac{\pi\mu}{\mu+\theta} \right) t^{2/(\mu+\theta)}, \quad (20) \end{aligned}$$

where we assumed $\delta = 2\mu/(\mu+\theta) < \mu$, which implies $\theta > 2 - \mu$. On that condition, the integral in Eq. (20) is convergent and the variance exists. We conclude that the diffusion process—in which the stochastic driving force is Lévy distributed and the stochastic equation is interpreted in the Stratonovich sense—may not be accelerated for the case without any external potential. If the variance exists, the diffusion is anomalously weak since the convergence condition coincides with the subdiffusion condition: the variance rises with time slower than linearly, $\langle x^2 \rangle \sim t^\delta$. The slope of that dependence diminishes with the parameter θ . In the case $\mu=2$, beside the

subdiffusion, also the enhanced diffusion occurs, for $\theta < 0$, as well as the normal one if $\theta=0$.

IV. LINEAR FORCE

In the case of stochastic motion in the harmonic oscillator field, $F(x) = -\lambda x$, which is governed by the Langevin equation with the Gaussian white noise (the Ornstein-Uhlenbeck process), the probability distribution converges with time to a steady state which corresponds to the Boltzmann equilibrium distribution. If the driving noise has the Lévy distribution with $\mu < 2$, the stationary limiting distribution still exist but the Boltzmann equilibrium is not reached and the variance is infinite [12]. We will demonstrate that, if the multiplicative noise is interpreted in the Stratonovich sense, the steady state may have the finite variance.

In the Itô interpretation, FPE is given by Eq. (6),

$$\frac{\partial}{\partial t} p_I(x, t) = \lambda \frac{\partial}{\partial x} [x p_I(x, t)] + K^\mu \frac{\partial^\mu}{\partial |x|^\mu} [x^{-\theta} p_I(x, t)], \quad (21)$$

and its Fourier transformation yields

$$\frac{\partial}{\partial t} \tilde{p}_I(k, t) = -\lambda k \frac{\partial}{\partial k} \tilde{p}_I(k, t) - K^\mu |k|^\mu \mathcal{F}_c [x^{-\theta} p_I(x, t)]. \quad (22)$$

The solution of Eq. (22) can be obtained [42] in a similar way as in the case $F(x)=0$, namely by inserting the expression (13) into Eq. (22). Then the Fourier transforms of the respective functions are expanded and terms of the order $|k|^{2\mu+\theta}$ and higher are neglected. The same conditions for the Fox function coefficients are required because contribution from the drift term contains only the component $|k|^\mu$. Therefore we obtain the solution of Eq. (21) in the form (15). The comparison of terms of the order $|k|^\mu$ on both sides of Eq. (22) results in a simple differential equation for the function $a(t)$. Its solution reads

$$a(t) = \left[\frac{\lambda/c_L}{1 - \exp[-\lambda(\mu + \theta)t]} \right]^{1/(\mu+\theta)}. \quad (23)$$

The constant $c_L = K^\mu h_0 / \mu h_\mu$ involves the expansion coefficients h_μ and h_0 of the functions \tilde{p}_I and $\mathcal{F}_c [x^{-\theta} p_I]$, which correspond to the orders $|k|^\mu$ and k^0 , respectively. They are given by $h_\mu = N(\mu + \theta) \Gamma(-\mu) \Gamma(1 + \mu + \theta) \cos(\pi\mu/2) / \Gamma[a_2 + A_2(1 + \mu)] \Gamma[-(\mu + \theta)/(2 + \theta)]$ and $h_0 = N(\mu + \theta) / (2 + \theta) \Gamma[a_2 + A_2(1 - \theta)]$, where the normalization constant $N = \Gamma[-\theta/(2 + \theta)] \Gamma[a_2 + A_2] / 2 \Gamma(1 + \theta) \Gamma[-\theta/(\mu + \theta)]$. The numerical values of the solution can be computed by means of series expansions, both for small and large $|x|$. Expansion of the function (15) in powers of $|x|^{-1}$ produces the following expression:

$$p_I(x, t) = N(\mu + \theta) \sum_{i=1}^{\infty} \frac{\Gamma[1 + (\mu + \theta)i]}{\Gamma(a_2 + A_2[1 - \theta + (\mu + \theta)i]) \Gamma\left(-\frac{\mu + \theta}{2 + \theta} i\right) i!} a(t)^{\theta - (\mu + \theta)i} |x|^{-1 + \theta - (\mu + \theta)i}, \quad (24)$$

where $a(t)$ is given by Eq. (23). In the following, we assume the remaining Fox function coefficients in the form: $a_2 = 1/2 + \theta(1 + \theta)/(2 + \theta)$ and $A_2 = 1 - 1/(2 + \theta)$. For these coefficients, $p_I(x, t)$ agrees with the exact diffusion equation solution in the case $\mu=2$: it corresponds to the stretched-Gaussian dependence [38]

$$p_I(x, t) \sim a^{1+\theta}|x|^\theta \exp[-\text{const}(a|x|)^{2+\theta}]. \quad (25)$$

Equation (24) implies the asymptotic shape of the distribution, $p_I(x, t) \sim |x|^{-1-\mu}$, the same as for the case $F(x)=0$. Therefore, the variance is also divergent for all θ and $\mu < 2$.

To obtain the solution of Eq. (1) in the Stratonovich interpretation, we introduce the variable y , which is given by Eq. (16), and transform the drift term according to Eq. (8). The Langevin equation takes the form

$$\dot{y} = -\frac{\lambda}{\mu}(\mu + \theta)y + \eta(t) \quad (26)$$

and FPE, expressed by the new variable, has the constant diffusion coefficient,

$$\frac{\partial}{\partial t} p_S(y, t) = \lambda(1 + \theta/\mu) \frac{\partial}{\partial y} [y p_S(y, t)] + \frac{\partial^\mu}{\partial |y|^\mu} p_S(y, t). \quad (27)$$

The above equation can be solved exactly with the initial condition $p_S(y, 0) = \delta(y)$ [12]. The Fourier transform of the solution reads $\tilde{p}_S(k, t) = \exp[-K^\mu \sigma(t) |k|^\mu]$, where

$$\sigma(t) = \frac{1 - \exp[-\lambda(\mu + \theta)t]}{\lambda(\mu + \theta)}. \quad (28)$$

After inverting the Fourier transform and transforming back to the original variable, we obtain the probability density distribution in the following form:

$$p_S(x, t) = \frac{\mu + \theta}{\mu^2 |x|} H_{2,2}^{1,1} \left[\frac{|x|^{1+\theta/\mu}}{(1 + \theta/\mu) K \sigma^{1/\mu}} \middle| \begin{matrix} (1, 1/\mu), (1, 1/2) \\ (1, 1), (1, 1/2) \end{matrix} \right], \quad (29)$$

which can be evaluated by series expansions, similar to Eq. (24). The tail of the distribution $p_S(x, t)$ has the same form as in the case $F(x)=0$: $p_S(x, t) \sim |x|^{-1-\mu-\theta}$. The second moment is convergent if $\theta > 2 - \mu$. On that condition, the system reaches with time a steady state which is characterized by the variance

$$\lim_{t \rightarrow \infty} \langle x^2 \rangle(t) = -\frac{2}{\pi\mu} \left[K \left(\frac{\theta}{\mu} + 1 \right) \right]^{2\mu/(\mu+\theta)} \Gamma \left(-\frac{2}{\mu + \theta} \right) \Gamma \left(1 + \frac{2\mu}{\mu + \theta} \right) \sin \left(\frac{\pi\mu}{\mu + \theta} \right) [(\mu + \theta)\lambda]^{-2/(\mu+\theta)}. \quad (30)$$

The above quantity is presented in Fig. 1 as a function of θ for some values of μ and λ . In all cases it declines with θ ; this fall is particularly rapid for large λ . Predominantly, the distribution is broader for smaller μ but this trend turns to the opposite in the limit $\theta \rightarrow \infty$. The parameter θ influences the convergence speed to the steady state, according to Eq.

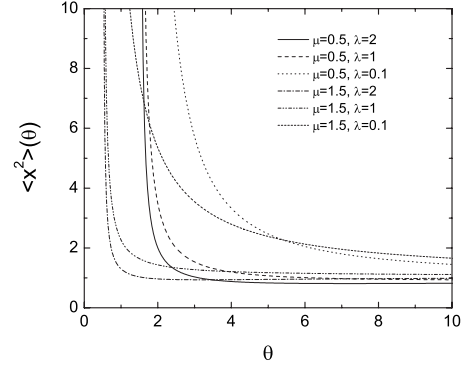


FIG. 1. Variance of the steady-state solution of Eq. (1) in the Stratonovich interpretation for the case $F(x) = -\lambda x$, calculated from Eq. (30).

(28), which is the same as for the Itô interpretation, cf. Equation (23).

V. NUMERICAL SIMULATIONS

In this section, we compare the analytical results with numerical stochastic trajectory simulations from the Langevin Eq. (1). For the Itô interpretation we apply the Euler method. Equation (2) implies that for each integration step τ the stochastic integral from the function $G(x)$ can be expressed by the noise value η_i by means of the following formula [43]:

$$\int_0^t G[x(t)] \eta dt = \int_0^t G[x(t)] d\eta = \sum_{i=0}^N G[x(t_i)] \tau^{1/\mu} \eta_i, \quad (31)$$

where $t_i = i\tau$ and $N = t/\tau$; the random numbers η_i are sampled from the Lévy distribution (4) with the order parameter μ . In the case of the Stratonovich interpretation, we first transform Eq. (1) to the corresponding equation with the additive noise. Then we apply the Heun method of integration,

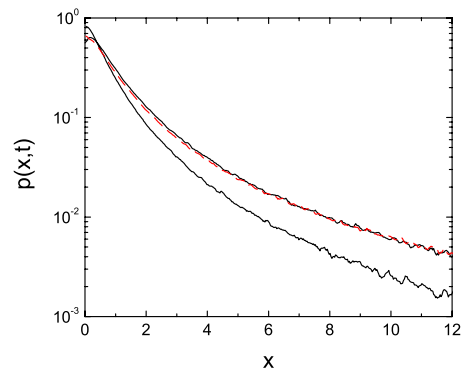


FIG. 2. (Color online) Comparison of numerical solutions of Eq. (1) for the case $G(x)=x$ and $F(x)=0$, with $\mu=1.5$ at $t=1$. The distribution $p_I(x, t)$ (lower line) was calculated according to Eq. (31) and $p_S(x, t)$ from Eq. (32) (upper line). The dashed line presents result of the evaluation of $p_S(x, t)$ by means of Eq. (3).

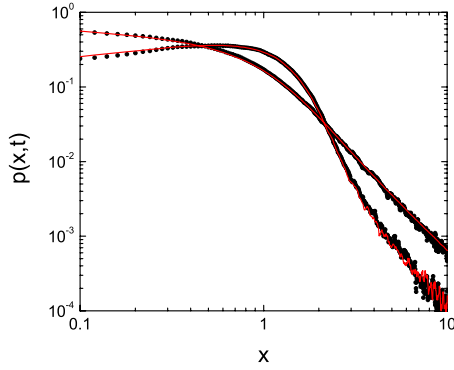


FIG. 3. (Color online) Comparison of numerical solutions of Eq. (1) for $G(x)=|x|^{-\theta\mu}$ and $F(x)=-\lambda x$, with parameters: $\mu=1.8$, $\lambda=1$, and $t=1$, in the Stratonovich interpretation. Distributions marked by lines were calculated by using the variable transformation, from Eq. (32), and those marked by dots—directly from Eq. (3). Cases for two values of θ are presented: $\theta=-0.2$ (upper line for small and large x) and $\theta=0.5$.

$$y_{i+1} = y_i + [\hat{F}[y(t_i)] + \hat{F}\{y(t_i) + \hat{F}[y(t_i)]\tau\}]\tau/2 + \tau^{1/\mu}\eta_i. \tag{32}$$

Transformation back to the original variable produces the stochastic trajectory $x(t)$. We demonstrate results of those algorithms in Fig. 2 for the simple case $G(x)=x$ and $F(x)=0$. The applied parameters, number of trajectories 10^6 and $\tau=0.001$, ensure sufficient accuracy. The above results are compared with the probability distribution obtained from the numerical integration of the equation with the multiplicative noise, in which the stochastic integral is defined by Eq. (3). Agreement of both results is very good. A similar comparison is presented in Fig. 3 for the nonlinear $G(x)$ in the form (11) with two values of the parameter θ , both positive and negative; no cutoff in the Lévy measure was introduced. This case is numerically more complicated because the difference formula is not explicit and numerical solving of a nonlinear

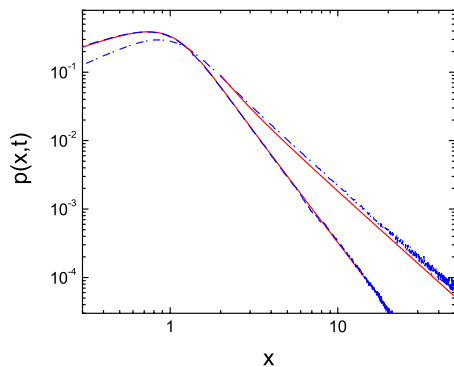


FIG. 4. (Color online) Probability distributions obtained from trajectory simulations for the case $F(x)=-\lambda x$. The Stratonovich ($p_S(x,t)$) and Itô [$p_I(x,t)$] results are marked by the dashed and dashed-dot lines, respectively. The analytical solutions of Eq. (1) (solid lines) follow from the series expansions of Eq. (29) both for small and large $|x|$, as well as from Eq. (24). Parameters are the following: $\mu=1.2$, $\theta=1$, $\lambda=1$, and $t=1$.

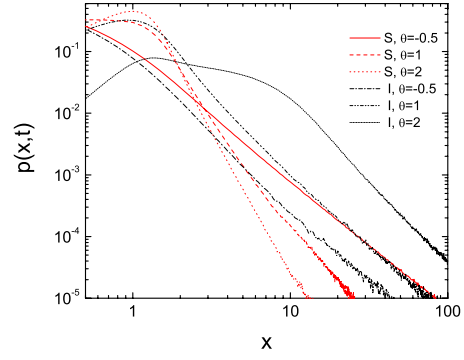


FIG. 5. (Color online) Probability distributions $p_S(x,t)$ and $p_I(x,t)$ obtained from trajectory simulations for the case $F(x)=-\lambda x$ at $t=1$, with $\mu=1.5$ and $\lambda=1$.

equation is required at each integration step. Agreement of both methods of calculation in Fig. 3 demonstrates that rules of the ordinary calculus are applicable for the Lévy processes with infinite variance.

The comparison with analytical results for the linear drift, presented in Fig. 4, indicate good agreement of those methods of solution. On the other hand, results which correspond to the Itô and Stratonovich interpretations for the same case differ considerably. Since they coincide for $\theta=0$, one can expect that the difference rises with $|\theta|$. In Fig. 5, the probability distributions for various θ , evaluated by numerical trajectory simulations, are presented. The difference between $p_I(x,t)$ and $p_S(x,t)$ for $\theta=2$ is indeed very large. The slope of $p_I(x,t)$ remains constant for a given μ and that of $p_S(x,t)$ rises with θ . We present those slopes, as a function of the order parameter μ , in Fig. 6. The slopes rise with μ , according to the analytical results $-\mu-1$ and $-\mu-\theta-1$ for the Itô and Stratonovich interpretations, respectively.

VI. SUMMARY AND CONCLUSIONS

We have studied the nonlinear Langevin equation with the multiplicative white noise which is distributed according to the Lévy statistics and has the power-law strength. The cor-

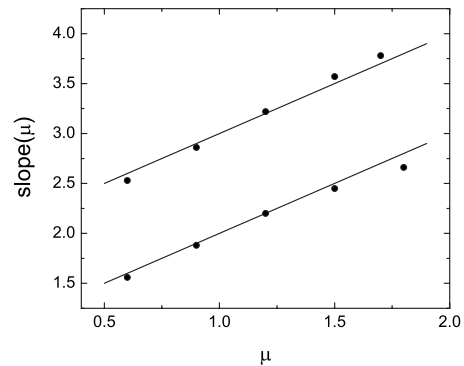


FIG. 6. Slope of the solution $p_I(x,t)$ (lower points) and of $p_S(x,t)$ (upper points) for large $|x|$, calculated by the numerical integration of Eq. (1) for the case $F(x)=-\lambda x$ and $t=1$. The parameters: $\lambda=1$ and $\theta=1$. The lower and upper straight lines have the slopes $-\mu-1$ and $-\mu-\theta-1$, respectively.

responding Fokker-Planck equation is fractional and its form depends on the interpretation of the stochastic equation. In the Itô interpretation, FPE possesses the variable diffusion coefficient. In the case without any external force, FPE coincides with the diffusion equation which was obtained in the framework of the coupled CTRW with the position-dependent waiting time. Solution of FPE with variable diffusion coefficient, both for the case without drift and for the harmonic oscillator potential, represents the Lévy process with simple scaling and the same order parameter as the driving noise. This property does not hold for other potentials. For example, solution to the problem of the wedge-shaped potential, $F(x) \sim \text{sgn } x$, studied in Ref [42], is a combination of two Lévy processes and simple scaling is lacking. Nevertheless, slowly decaying Lévy tails are present in that solution and then the variance is divergent for all θ .

In the Stratonovich interpretation, FPE has been obtained by the variable transformation; in this case the problem is reduced to Langevin equation with the additive noise. The resulting probability distributions differ considerably from those in the Itô sense. In particular, shape of the tail depends on the parameter of noise intensity θ and, as a result, the variance may be convergent. Therefore the diffusion process, for the case without drift, can be either accelerated or anomalously weak, in contrast to the Itô result, which predicts only

the accelerated diffusion. For the case $\mu=2$, both approaches differ by an additional, effective force in the Langevin equation. The disagreement between both interpretations is in fact not surprising since the deterministic force in the Langevin equation is not just the Newtonian one; those forces are identical only in the case of the external noise, i.e., for the Stratonovich interpretation [24]. Since the difference between the interpretations is deterministic in nature no different underlying stochastic dynamics is implied. For the general Lévy stable processes, relation between distributions in both interpretations is more complicated and the corresponding equations cannot be related one to the other by means of a drift term.

The distribution tails, which are algebraic and fall rapidly enough to ensure convergence of the variance (“fat tails”), are of physical importance. They are well known, e.g., in the field of the economic research [44,45]. Based on the poor empirical performance of the Black-Scholes model of option pricing, which mathematically resolves itself to the Langevin equation, one proposes to replace the Gaussian noise by the a Lévy one but with a truncated tail [18]. Such process may converge to the Gaussian so slowly, that numerical calculations yield only fat tails [38,46]. The Langevin equation with the multiplicative Lévy noise in the Stratonovich interpretation could be an alternative model of the fat tails.

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